UDC 534

DEVIATION OF THE SOLUTION OF A QUASI-LINEAR WAVE EQUATION FROM THE SOLUTION OF A LINEAR EQUATION IN THE DOMAIN OF CONTINUOUS FIRST DERIVATIVES

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We consider one-dimensional transient wave processes which under zero initial conditions are excited by a boundary force and are described by a quasi-linear wave equation of general form. Conditions are imposed on the boundary force such that at the initial stage of the process a domain of continuous first derivatives exists. For the successive approximation of the solution of the quasi-linear equation in this domain we propose a procedure in which the solution of a linear homogeneous wave equation serves as the zeroth approximation, while the succeeding approximations are computed by integrating the inhomogeneous wave equations obtained from the original quasi-linear equation by approximating the nonlinear terms by means of the preceding approximation. We consider the application of this procedure for constructing the asymptotic approximations and we analyze the deviation of the nonlinear solution from the linear solution (the zeroth approximation) as a function of the coefficients of the quasi-linear equation and of the nature of the boundary force. As an illustration we examine geometrically and physically the transient wave processes of deformation of an elastic halfspace. We show that in the special case of an abruptly applied force, which subsequently varies sinusoidally with time, the nonlinear effects lead not only to a variation in the amplitude of the linear solution, but also to the appearance of qualitiatively different high-frequency components of the solution. The approximation procedure which in the present paper has been proposed, by example of a second-order quasi-linear equation, for the construction of a solution of the travelling wave type, is related in concept, to a certain extent, to the method of perturbations [1]. We remark that the procedure of successive approximation was applied in [2] for constructing the solution of a second-order quasi-linear equation in the form of an expansion in standing waves. To some extent, closely related to the present paper are the investigations in [3 – 5] in which the dynamic process, modelled by a quasi-linear system of equations, is described approximately as the sum of two components of which one is determined as the solution of the linear wave equation, while the other is constructed in a non-wave form by the method of perturbations.

1. Statement of the problem. Let ξ be a dimensionless coordinate, τ dimensionless time, $u\ (\xi,\ \tau)$ the unknown function, ε a small positive number and $H\ (\tau)$ the Heaviside function. Let a prime denote the derivative with respect to ξ, while a dot – the derivative with respect to τ. In the region $\xi \geqslant 0,\ \tau \geqslant 0$ we consider the integration of the quasi-linear equation

$$u^{\cdot\cdot}(\xi,\tau)p(u^{\cdot},u',u;\xi,\tau)-u''(\xi,\tau)q(u^{\cdot},u',u;\xi,\tau)=\\=R(u^{\cdot},u',u;\xi,\tau) \tag{1.1}$$

with coefficients

$$p(u^{\cdot},u',u;\xi,\tau)=1+P(u^{\cdot},u',u;\xi,\tau)\\ q(u^{\cdot},u',u;\xi,\tau)=1+Q(u^{\cdot},u',u;\xi,\tau) \tag{1.2}$$

$$P(u^{\cdot},u',u;\xi,\tau)=a_2u^{\cdot}+a_1u'+a_0u+a_{22}(u^{\cdot})^2+\\ a_{21}u^{\cdot}u'+a_{11}(u')^2+a_{20}u^{\cdot}u+a_{10}u'u+a_{00}u^2+\ldots \tag{1.3}$$

$$Q(u^{\cdot},u',u;\xi,\tau)=b_2u^{\cdot}+b_1u'+b_0u+b_{22}(u^{\cdot})^2+b_{21}u^{\cdot}u'+\\ b_{11}(u')^2+b_{20}u^{\cdot}u+b_{10}u'u+b_{00}u^2+\ldots \tag{1.4}$$

$$R(u^{\cdot},u',u;\xi,\tau)=c_{22}(u^{\cdot})^2+c_{21}u^{\cdot}u'+c_{11}(u')^2+c_{20}u^{\cdot}u+\\ c_{10}u'u+c_{00}u^2+c_{222}(u^{\cdot})^3+c_{221}(u^{\cdot})^2u'+c_{211}u^{\cdot}(u')^2+\\ c_{111}(u')^3+c_{220}(u^{\cdot})^2u+c_{200}u^{\cdot}u^2+c_{110}(u')^2u+\\ c_{100}u'u^2+c_{210}u^{\cdot}u'u+c_{000}u^3+\ldots \tag{1.5}$$

Here $a_i, a_{ij}, \ldots, b_i, b_{ij}, \ldots, c_{ij}, c_{ijh}, \ldots$ are continuous functions of ξ and τ, which acquire finite values. Here the number of indices 0 shows the power of $u(\xi,\tau)$, the number of indices 1 shows the power of $u'(\xi,\tau)$, and the number of indices 2 shows the power of $u^{\cdot}(\xi,\tau)$ in the term standing after the coefficient.

We give the initial conditions

$$u(\xi,0)=0,\qquad u^{\cdot}(\xi,0)=0 \tag{1.6}$$

one of the following boundary conditions:

$$u'(0,\tau)=\varepsilon\Psi(\tau)H(\tau),\quad \text{(Problem } A) \tag{1.7}$$

$$u^{\cdot}(0,\tau)=-\varepsilon\Psi(\tau)H(\tau)\quad \text{(Problem } B) \tag{1.8}$$

and the condition of damping at infinity

$$u(\infty,\tau)=0 \tag{1.9}$$

If the function $u(0,\tau)$ were given for $\xi=0$, then by a differentiation of $u(0,\tau)$ with respect to τ the problem can be reduced to case (1.8). We require that the specified function $\Psi(\tau)$ have, for $\tau\geqslant 0$, finite continuous derivatives of all the orders encountered in the subsequent discussions and that it satisfies the conditions

$$\Psi(0)=0,\qquad \max|\Psi(\tau)|\leqslant 1\quad \text{for}\ \ \tau>0 \tag{1.10}$$

We are easily convinced that in the problem statement adopted, as a function of the coefficients of representations (1.3)–(1.5) and of the properties of function $\Psi(\tau)$, either for any $\xi\geqslant 0$ and $\tau\geqslant 0$, or in some finite time interval $0\leqslant\tau\leqslant\tau_1=\text{const}$, $0\leqslant\xi\leqslant\tau$, the conditions

$$p(u^{\cdot},u',u;\xi,\tau)>0,\qquad q(u^{\cdot},u',u;\xi,\tau)>0 \tag{1.11}$$

are fulfilled and Eq. (1.1), being hyperbolic, has a travelling wave type solution. Here either for any $\xi\geqslant 0$ and $\tau\geqslant 0$ or at some initial stage $0\leqslant\tau\leqslant\tau_0=\text{const}$

of the wave process, $u^{\cdot}(\xi, \tau)$, $u'(\xi, \tau)$ and $u(\xi, \tau)$ are continuous functions. We note that $u^{\cdot}(\xi, \tau)$ and $u'(\xi, \tau)$ become discontinuous at a finite value of time $\tau = \tau_0$ when the coefficients of representations (1.3) – (1.5) and the function $\Psi(\tau)$ are such that a shock wave occurs at $\tau = \tau_0$ [6 – 9].

A number of nonlinear problems of mechanics and accoustics lead to the integration of special forms of Eq. (1.1) with coefficients of (1.2) – (1.5) type. For example, the nonlinear posing of the problems of one-dimensional transient wave processes of the deformation of an elastic half-space [6 – 9] and of elastic rods [10, 11] leads to the special case $P = 0$, $R = 0$ and $Q = Q(u')$ with constant coefficients b_i, $b_{ij}, \ldots$ This special case is analyzed in Sects. 5 and 6.

2. The successive approximation procedure. Let us consider the successive approximation of a wave solution of Eq. (1.1) for small values of τ in the domain of continuous $u^{\cdot}(\xi, \tau)$, $u'(\xi, \tau)$ and $u(\xi, \tau)$, using a procedure in which the zeroth approximation $(j = 0)$ is determined by integrating the linear homogeneous equation

$$u_0^{\cdot\cdot}(\xi, \tau) - u_0''(\xi, \tau) = 0 \tag{2.1}$$

while the succeeding approximations $(j = 1, 2, \ldots)$ are determined by integrating the linear inhomogeneous equations

$$u_j^{\cdot\cdot}(\xi, \tau) - u_j''(\xi, \tau) = G_j(\xi, \tau) \qquad (j = 1, 2, \ldots) \tag{2.2}$$

where

$$G_j(\xi, \tau) = -\ddot{u}_{j-1}(\xi, \tau) P(\dot{u}_{j-1}, u'_{j-1}, u_{j-1}; \xi, \tau) + u''_{j-1}(\xi, \tau) Q(\dot{u}_{j-1}, u'_{j-1}, u_{j-1}; \xi, \tau) + R(\dot{u}_{j-1}, u'_{j-1}, u_{j-1}; \xi, \tau) \tag{2.3}$$

The zeroth approximation of the solution of Problem A, i.e. the solution of the linear wave equation (2.1) under the boundary conditions (1.6), (1.7), (1.9), and the zeroth approximation of the solution of Problem B, i.e. the solution of the linear wave equation (2.1) under the boundary conditions (1.6), (1.8), (1.9), are the same and can be represented in the following form:

$$u_0(\xi, \tau) = \int_0^{\tau} u_0^{\cdot}(\xi, t)\, dt = -\varepsilon \Psi_1(\tau - \xi) H(\tau - \xi) \tag{2.4}$$

$$u_0'(\xi, \tau) = -u_0^{\cdot}(\xi, \tau) = \varepsilon \Psi(\tau - \xi) H(\tau - \xi) \tag{2.5}$$

$$u_0''(\xi, \tau) = u_0^{\cdot\cdot}(\xi, \tau) = -\varepsilon \Psi(\tau - \xi) H(\tau - \xi) \tag{2.6}$$

Here and further

$$\Psi_1(\tau - \xi) = \int_0^{\tau - \xi} \Psi(z)\, dz \tag{2.7}$$

We can convince ourselves that in the domain of continuous $u^{\cdot}(\xi, \tau)$, $u'(\xi, \tau)$ and $u(\xi, \tau)$, when computing the succeeding approximations $j = 1, 2, 3, \ldots$ the right-hand sides of Eqs. (2.2) have the structure

$$G_j(\xi, \tau) = g_j(\xi, \tau) H(\tau - \xi) \tag{2.8}$$

where $g_j(\xi, \tau)$ are continuous functions when $\tau \geqslant \xi$.

Thus, the computation of approximations $j = 1, 2, 3, \ldots$ reduces to the integration of the inhomogeneous linear wave equations

$$u_j^{\cdot\cdot}(\xi,\tau)-u_j''(\xi,\tau)=g_j(\xi,\tau)H(\tau-\xi) \tag{2.9}$$

Using the Laplace transform and applying the method used in [12], we can show that the exact solution of Eq. (2.9) in the case of Problems A and B can be represented as

$$u_j(\xi,\tau)=\int_0^\tau u_j^{\cdot}(\xi,t)\,dt \tag{2.10}$$

$$\begin{aligned}
u_j'(\xi,\tau)&=\varepsilon\Psi(\tau-\xi)H(\tau-\xi)-{}^1/_2TF_{1j}(\xi,\tau)-{}^1/_2F_{2j}(\xi,\tau)+{}^1/_2F_{3j}(\xi,\tau)\\
u_j^{\cdot}(\xi,\tau)&=-\varepsilon\Psi(\tau-\xi)H(\tau-\xi)+{}^1/_2TF_{1j}(\xi,\tau)+{}^1/_2F_{2j}(\xi,\tau)+{}^1/_2F_{3j}(\xi,\tau)
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
u_j''(\xi,\tau)=\Big\{&-\varepsilon\Psi^{\cdot}(\tau-\xi)-g_j(\xi,\tau)+\frac{1}{4}Tg_j\left(\frac{\tau-\xi}{2},\ \frac{\tau-\xi}{2}\right)+\\
&\frac{1}{4}g_j\left(\frac{\tau+\xi}{2},\ \frac{\tau+\xi}{2}\right)\Big\}H(\tau-\xi)+\\
&\frac{1}{2}TF_{1j}^*(\xi,\tau)+\frac{1}{2}F_{2j}^*(\xi,\tau)+\frac{1}{2}F_{3j}^*(\xi,\tau)\\
u_j^{\cdot\cdot}(\xi,\tau)=u_j''&(\xi,\tau)+g_j(\xi,\tau)H(\tau-\xi)
\end{aligned} \tag{2.12}$$

Here $T=1$ in the case of Problem A and $T=-1$ in the case of Problem B. In (2.11) and (2.12) we have used notation (2.13) and (2.14), respectively

$$\begin{aligned}
F_{1j}(\xi,\tau)&=H(\tau-\xi)\int_0^{(\tau-\xi)/2}g_j(x,y_1)\,dx, \quad y_1=\tau-\xi-x\\
F_{2j}(\xi,\tau)&=H(\tau-\xi)\int_0^{\xi}g_j(x,y_2)\,dx, \quad y_2=\tau-\xi+x\\
F_{3j}(\xi,\tau)&=H(\tau-\xi)\int_\xi^{(\tau+\xi)/2}g_j(x,y_3)\,dx, \quad y_3=\tau+\xi-x
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
F_{1j}^*(\xi,\tau)&=H(\tau-\xi)\int_0^{(\tau-\xi)/2}g_j^{\cdot}(x,y_1)\,dx, \quad y_1=\tau-\xi-x\\
F_{2j}^*(\xi,\tau)&=H(\tau-\xi)\int_0^{\xi}g_j^{\cdot}(x,y_2)\,dx, \quad y_2=\tau-\xi+x\\
F_{3j}^*(\xi,\tau)&=H(\tau-\xi)\int_\xi^{(\tau+\xi)/2}g_j^{\cdot}(x,y_3)\,dx, \quad y_3=\tau+\xi-x
\end{aligned} \tag{2.14}$$

For concretely specified function $\Psi(\tau)$ and coefficients $a_i, a_{ij}, \ldots, b_i, b_{ij}, \ldots, c_{ij}, c_{ijh}, \ldots$ by an analytic or numerical computation of the integrals occurring in formulas (2.13) and (2.14) we can find the several first approximations ($j=1,2,\ldots$) of the wave solution of the quasi-linear equation (1.1) under the boundary conditions of Problems A and B. From these approximations we can establish how the solution of Eq. (1.1) differs from the solution of linear equation (2.1) with increasing time. The approximation precedure formulated can be applied also in the case of functions $P(u^{\cdot}, u', u;\ \xi,\ \tau)$, $Q(u^{\cdot}, u', u;\ \xi,\ \tau)$ and $R(u^{\cdot},\ u', u;\ \xi,\ \tau)$ which differ from (1.3) – (1.5) but which for sufficiently small values of τ satisfy the conditions

$$|P| < 1, \qquad |Q| < 1$$
$$|R| < |u^{\cdot\cdot}(\xi, \tau)|, \qquad |B| < |u''(\xi, \tau)|$$

3. Realisation of the procedure in the form of an asymptotic approximation. The approximation procedure described can be used to construct asymptotic approximations as $\varepsilon \to 0$. To do this we should compute $g_1(\xi, \tau)$ to within terms with the factor ε^2, $g_2(\xi, \tau)$ to within terms with the factors ε^2, ε^3, etc. For such accuracy of computation the functions $g_j(\xi, \tau)$ have the structure

$$g_j(\xi,\tau) = \varepsilon \sum_{i=1}^{j} \varepsilon^i g_i^*(\xi, \tau) \tag{3.1}$$

where $g_i^*(\xi, \tau)$ does not depend upon ε. The asymptotic approximations of the unknown functions $u_j(\xi, \tau)$ and of their derivatives are subject to computation by the general formulas (2.10) – (2.14). For such a realization of the procedure the jth approximation has the structure

$$u_j(\xi, \tau) = \varepsilon H(\tau - \xi) \sum_{i=0}^{j} \varepsilon^i v_i(\xi, \tau) \tag{3.2}$$

where $v_i(\xi, \tau)$ does not depend upon ε.

In view of the fact that functions (1.3) – (1.5) are polynomials in $u^{\cdot}(\xi, \tau)$, $u'(\xi, \tau)$ and $u(\xi, \tau)$, the realization of the proposed approximate procedure as asymptotic approximations, as $\varepsilon \to 0$, yields a solution in the form of sum (3.2) which in structure is analogous to the original assumption used when applying the method of perturbations [1]. However, as far as the author is aware, the transient wave processes considered in the present paper have not been investigated by the method of perturbations. We remark that to within representation (3.1) function $g_1(\xi, \tau)$ and, correspondingly, also the first asymptotic approximation $u_1(\xi, \tau)$ are defined by the function $\Psi(\tau)$ and by the coefficients a_i, b_i, c_{ij} $(i, j = 0, 1, 2)$ of representations (1.3) – (1.5). The second asymptotic approximation $u_2(\xi, \tau)$ depends on $\Psi(\tau)$ and on the coefficients $a_i, a_{ij}, b_i, b_{ij}, c_{ij}, c_{ijh}$ $(i, j, h = 0, 1, 2)$ of representations (1.3) – (1.5). The coefficients of representations (1.3) – (1.5), on which the succeeding asymptotic approximations depend can be indicated by analogy.

Let us consider further the case of constant coefficients of representations (1.3)–(1.5) and for this case give explicit formulas for the first two asymptotic approximations. From what is set forth below we see that in the case mentioned the functions $v_i(\xi, \tau)$ in (3.2) have the structure

$$v_i(\xi, \tau) = \sum_{k=0}^{i} \xi^k \eta_{ik}(\tau - \xi) \tag{3.3}$$

and in the jth approximation the asymptotic solutions of Problems A and B for $\varepsilon \to 0$, can be written in the form

$$u_j(\xi, \tau) = u_0(\tau - \xi) + H(\tau - \xi)\, \varepsilon \sum_{i=1}^{j} \varepsilon^i \sum_{k=0}^{i} \xi^k \eta_{ik}(\tau - \xi)$$

$$u_j'(\xi, \tau) = u_0'(\tau - \xi) + H(\tau - \xi)\, \varepsilon \sum_{i=1}^{j} \varepsilon^i \sum_{k=0}^{i} \xi^k \eta_{\xi ik}(\tau - \xi)$$

$$u_j^{\cdot}(\xi,\tau)=u_0^{\cdot}(\tau-\xi)+H(\tau-\xi)\,\varepsilon\sum_{i=1}^{j}\varepsilon^i\sum_{k=0}^{i}\xi^k\eta_{\tau ik}(\tau-\xi)$$
$$u_j''(\xi,\tau)=u_0''(\tau-\xi)+H(\tau-\xi)\,\varepsilon\sum_{i=1}^{j}\varepsilon^i\sum_{k=0}^{i}\xi^k\eta_{\xi\xi ik}(\tau-\xi) \qquad (3.4)$$
$$u_j^{\cdot\cdot}(\xi,\tau)=u_0^{\cdot\cdot}(\tau-\xi)+H(\tau-\xi)\,\varepsilon\sum_{i=1}^{j}\varepsilon^i\sum_{k=0}^{i}\xi^k\eta_{\tau\tau ik}(\tau-\xi)$$

Here η_{ik}, $\eta_{\xi ik}$, $\eta_{\tau ik}$, $\eta_{\xi\xi ik}$ and $\eta_{\tau\tau ik}$ are functions which depend on ξ and τ in terms of the difference $\tau-\xi$. However, the forms of these functions for Problems A and B are different and are determined by the function $\Psi(\tau)$ and by the coefficients of representations (1.3) – (1.5).

The first asymptotic approximation in the case of constant coefficients of representations (1.3) – (1.5). Having carried out the calculations for $j=1$ on the basis of formulas (1.3) – (1.5) and (2.3) – (2.8), we have the asymptotic approximation

$$g_1(\xi,\tau)=g_{10}(\tau-\xi),\qquad g_{10}(\tau-\xi)=\varepsilon^2 w_1(\tau-\xi) \qquad (3.5)$$

Here

$$w_1(\tau-\xi)=(A_1-A_2)\,\Psi^{\cdot}(\tau-\xi)\,\Psi(\tau-\xi)-A_0\Psi^{\cdot}(\tau-\xi)\Psi_1(\tau-\xi)+$$
$$C_2\Psi^2(\tau-\xi)+C_1\Psi(\tau-\xi)\Psi_1(\tau-\xi)+C_0\,\Psi_1^2(\tau-\xi) \qquad (3.6)$$

$$A_2=a_2-b_2,\quad A_1=a_1-b_1,\quad A_0=a_0-b_0$$
$$C_2=c_{22}-c_{21}+c_{11},\quad C_1=c_{20}-c_{10},\quad C_0=c_{00} \qquad (3.7)$$

and we have used definition (2.7). Due to the fact that in the case being considerd $g_1(\xi,\tau)=g_{10}(\tau-\xi)$ and $g_{10}(0)=0$, the general formulas (2.11) and (2.12) simplify into formulas (3.8) and (3.9), respectively,

$$u_1'(\xi,\tau)=\left\{\varepsilon\Psi(\tau-\xi)+\frac{1}{4}(1-T)\int_0^{\tau-\xi}g_{10}(z)\,dz-\frac{1}{2}\xi g_{10}(\tau-\xi)\right\}H(\tau-\xi)$$

$$u_1^{\cdot}(\xi,\tau)=\left\{-\varepsilon\Psi(\tau-\xi)+\frac{1}{4}(1+T)\int_0^{\tau-\xi}g_{10}(z)\,dz+\right. \qquad (3.8)$$
$$\left.\frac{1}{2}\xi g_{10}(\tau-\xi)\right\}H(\tau-\xi)$$

$$u_1''(\xi,\tau)=\{-\varepsilon\Psi^{\cdot}(\tau-\xi)+{}^1\!/_4\,(T-3)\,g_{10}(\tau-\xi)+$$
$${}^1\!/_2\,\xi\,g_{10}^{\cdot}(\tau-\xi)\}\,H(\tau-\xi) \qquad (3.9)$$
$$u_1^{\cdot\cdot}(\xi,\tau)=u_1''(\xi,\tau)+g_{10}(\tau-\xi)\,H(\tau-\xi)$$

Substituting (3.5), (3.6) into (3.8) and (3.9), we obtain formulas of form (3.4), where in the given case $j=1$ and in the right-hand sides the first terms are the linear solution (2.4) – (2.6), while the second terms are defined in therms of the functions

$$\eta_{10}(\tau-\xi)=\frac{1}{4}(1+T)\int_0^{\tau-\xi}dz\int_0^z w_1(l)\,dl,\quad \eta_{11}(\tau-\xi)=\frac{1}{2}\int_0^{\tau-\xi}w_1(z)\,dz$$

$$\eta_{\xi10}(\tau-\xi)=\frac{1}{4}(1-T)\int_0^{\tau-\xi}w_1(z)\,dz,\quad \eta_{\xi11}(\tau-\xi)=-\frac{1}{2}w_1(\tau-\xi)$$

$$\eta_{\tau10}(\tau-\xi)=\frac{1}{4}(1+T)\int_0^{\tau-\xi}w_1(z)\,dz,\quad \eta_{\tau11}(\tau-\xi)=\frac{1}{2}w_1(\tau-\xi)$$

$$\eta_{\xi\xi10}(\tau-\xi)={}^1/_4(T-3)\,w_1(\tau-\xi),\quad \eta_{\xi\xi11}(\tau-\xi)={}^1/_2\,w_1^{\bullet}(\tau-\xi)$$

$$\eta_{\tau\tau10}(\tau-\xi)={}^1/_4(T+1)\,w_1(\tau-\xi),\quad \eta_{\tau\tau11}(\tau-\xi)={}^1/_2 w_1^{\bullet}(\tau-\xi)\qquad(3.10)$$

Thus, if function $\Psi(\tau)$ and the numerical values of coefficients a_i, b_i, c_{ij} $(i, j = 0, 1, 2)$ are given, then the problem of computing the first asymptotic approximation consists in a successive application of formulas (3.4), (2.4) – (2.6) and (3.10).

The second asymptotic approximation with constant coefficients of representation (1.3) – (1.5). Substituting the first asymptotic approximation into (1.3) – (1.5) and using (2.3) and (2.8), we have the asymptotic approximation

$$g_2(\xi,\tau)=g_{20}(\tau-\xi)+\xi g_{21}(\tau-\xi)\qquad(3.11)$$

Here

$$g_{20}(\tau-\xi)=\varepsilon^2 w_1(\tau-\xi)+\varepsilon^3 w_2(\tau-\xi),\quad g_{21}(\tau-\xi)=\varepsilon^2 f_2(\tau-\xi)\qquad(3.12)$$

In (3.11) the function $w_1(\tau-\xi)$ is defined by formula (3.6), while the functions $w_2(\tau-\xi)$ and $f_2(\tau-\xi)$ have the following values:

$$\begin{aligned}w_2(\tau-\xi)={}&\Psi^{\bullet}(\tau-\xi)\Big\{B_2\int_0^{\tau-\xi}w_1(z)\,dz+B_0\int_0^{\tau-\xi}dz\int_0^z w_1(l)\,dl+\\&A_{12}\Psi^2(\tau-\xi)+A_{11}\Psi(\tau-\xi)\Psi_1(\tau-\xi)+A_{10}\Psi_1^2(\tau-\xi)\Big\}+w_1(\tau-\\&\xi)\{B_1\Psi(\tau-\xi)+(B_0+b_0)\Psi_1(\tau-\xi)\}-\Big[C_2^*\Psi(\tau-\\&\xi)+C_1^*\Psi_1(\tau-\xi)\Big]\int_0^{\tau-\xi}w_1(z)\,dz-\frac{1}{2}(1+T)\Big[\frac{1}{2}C_1\Psi(\tau-\xi)+C_2\Psi_1(\tau-\\&\xi)\Big]\int_0^{\tau-\xi}dz\int_0^z w_1(l)\,dl-C_{13}\Psi^3(\tau-\xi)-C_{12}\Psi^2(\tau-\xi)\Psi_1(\tau-\xi)-C_{11}\Psi(\tau-\xi)\times\\&\Psi_1^2(\tau-\xi)-C_{10}\Psi_1^3(\tau-\xi)\end{aligned}\qquad(3.13)$$

$$\begin{aligned}f_2(\tau-\xi)={}&\frac{1}{2}\Psi^{\bullet}(\tau-\xi)\Big\{(A_2-A_1)w_1(\tau-\xi)+A_0\int_0^{\tau-\xi}w_1(z)\,dz\Big\}+\\&\frac{1}{2}w_1^{\bullet}(\tau-\xi)\{(A_2-A_1)\Psi(\tau-\xi)+A_0\Psi_1(\tau-\xi)\}-\\&w_1(\tau-\xi)\Big\{C_2\Psi(\tau-\xi)+\frac{1}{2}C_1\Psi(\tau-\xi)\Big\}-\Big[\frac{1}{2}C_1\Psi(\tau-\xi)+\\&C_0\Psi_1(\tau-\xi)\Big]\int_0^{\tau-\xi}w_1(z)\,dz\end{aligned}$$

In (3.13), in addition to (3.7) we have used further the following short notation for the constants:

$$A_{12} = a_{22} - b_{22} - a_{21} + b_{21} + a_{11} - b_{11}$$

$$A_{11} = a_{20} - b_{20} - a_{10} + b_{10}, \quad A_{10} = a_{00} - b_{00}$$

$$B_2 = {}^1/_4\,(1 + T) A_2 + {}^1/_4\,(1 - T)\,A_1$$

$$B_1 = {}^1/_4\,(1 + T)\,(A_2 - A_1) + b_2 - b_1, \quad B_0 = {}^1/_4\,(1 + T)\,A_0$$

$$C_{13} = c_{222} - c_{221} + c_{211} - c_{111}, \quad C_{12} = c_{220} - c_{210} + c_{110}$$

$$C_{11} = c_{200} - c_{100}, \quad C_{10} = c_{00}, \quad C_1{}^* = {}^1/_4\,(1 + T)c_{20} + {}^1/_4\,(1 - T)c_{10}$$

$$C_2{}^* = {}^1/_2\,(1 + T)c_{22} - {}^1/_2\,Tc_{21} + {}^1/_2\,(T - 1)c_{11}$$

In the special case being considered of a function $g_2\,(\xi,\ \tau)$ of form (3.11), possessing the property $g_{20}\,(0) = 0,\ f_2\,(0) = 0$, the general formulas (2.11) can be reduced to the form

$$u_2'\,(\xi, \tau) = \varepsilon\Psi\,(\tau - \varepsilon)\,H\,(\tau - \xi) + \frac{1}{4}(1 - T)\,H\,(\tau - \xi)\Big\{ \int\limits_0^{\tau-\xi} g_{20}(z)\,dz + \frac{1}{2}\int\limits_0^{\tau-\xi} dz \int\limits_0^{z} g_{21}\,(l)\,dl - \frac{1}{2}\xi H\,(\tau - \xi)\Big\{ g_{20}(\tau - \xi) - \frac{1}{2}\int\limits_0^{\tau-\xi} g_{21}\,(z)\,dz \Big\} - \frac{1}{4}\,\xi^2 g_{21}(\tau - \xi)\,H\,(\tau - \xi)$$

$$u_2^{\boldsymbol{\cdot}}\,(\xi, \tau) = -\,\varepsilon\Psi\,(\tau - \xi)\,H\,(\tau - \xi) + \frac{1}{4}(1 + T)\,H\,(\tau - \xi)\Big\{ \int\limits_0^{\tau-\xi} g_{20}(z)\,dz + \frac{1}{2}\int\limits_0^{\tau-\xi} dz \int\limits_0^{z} g_{21}\,(l)\,dl \Big\} + \frac{1}{2}\xi H\,(\tau - \xi)\Big\{ g_{20}(\tau - \xi) + \frac{1}{2}\int\limits_0^{\tau-\xi} g_{21}\,(z)\,dz \Big\} + \frac{1}{4}\,\xi^2 g_{21}(\tau - \xi)\,H\,(\tau - \xi)$$

Analogously we can modify formulas (2.12). By substitution (3.12) we further easily represent the solution in the form (3.4) where in the given case $j = 2$ and in the right-hand side there occur the functions (3.10) as well as the following functions:

$$\eta_{2k}\,(\tau - \xi) = \int\limits_0^{\tau-\xi} \eta_{\tau 2k}\,(z)\,dz \qquad (k = 0, 1, 2)$$

$$\eta_{\xi 20}\,(\tau - \xi) = \frac{1}{4}(1 - T)\Big\{ \int\limits_0^{\tau-\xi} w_2\,(z)\,dz + \frac{1}{2}\int\limits_0^{\tau-\xi} dz \int\limits_0^{z} f_2\,(l)\,dl \Big\}$$

$$\eta_{\xi 21}\,(\tau - \xi) = -\frac{1}{2}\,w_2\,(\tau - \xi) + \frac{1}{4}\int\limits_0^{\tau-\xi} f_2\,(z)\,dz, \quad \eta_{\xi 22}\,(\tau - \xi) = -\frac{1}{4}\,f_2(\tau - \xi)$$

$$\eta_{\tau 20}\,(\tau - \xi) = \frac{1}{4}(1 + T)\Big\{ \int\limits_0^{\tau-\xi} w_2\,(z)\,dz + \frac{1}{2}\int\limits_0^{\tau-\xi} dz \int\limits_0^{z} f_2\,(l)\,dl \Big\}$$

$$\eta_{\tau 21}\,(\tau - \xi) = \frac{1}{2}\,w_2\,(\tau - \xi) + \frac{1}{4}\int\limits_0^{\tau-\xi} f_2\,(z)\,dz, \quad \eta_{\tau 22}\,(\tau - \xi) = \frac{1}{4}\,f_2\,(\tau - \xi)$$

$$\eta_{\xi\xi 20}(\tau-\xi)=\frac{1}{4}(T-3)\,w_2(\tau-\xi)+\frac{1}{8}(1+T)\int_0^{\tau-\xi} f_2(z)\,dz$$

$$\eta_{\xi\xi 21}(\tau-\xi)={}^1/_2 w_2^{\cdot}(\tau-\xi)-{}^3/_4 f_2(\tau-\xi),\qquad \eta_{\xi\xi 22}(\tau-\xi)={}^1/_4 f_2^{\cdot}(\tau-\xi)$$

$$\eta_{\tau\tau 20}(\tau-\xi)=\frac{1}{4}(1+T)\,w_2(\tau-\xi)+\frac{1}{8}(1+T)\int_0^{\tau-\xi} f_2(z)\,dz$$

$$\eta_{\tau\tau 21}(\tau-\xi)={}^1/_2 w_2^{\cdot}(\tau-\xi)+{}^1/_4 f_2(\tau-\xi),\qquad \eta_{\tau\tau 22}(\tau-\xi)={}^1/_4 f_2^{\cdot}(\tau-\xi)$$

The second asymptotic approximation, as the first one, exactly satisfies the boundary conditions (1.6), (1.7), (1.9) of Problem A with $T=1$ and the boundary conditions (1.6), (1.8), (1.9) of Problem B with $T=-1$. As τ increases the perturbed domain $0\leqslant\xi\leqslant\tau$ grows. The stated formulas show how the solution of the quasi-linear equation (1.1) differs from the solution of linear equation (2.1) in the domain of continuous $u^{\cdot}(\xi,\tau)$, $u'(\xi,\tau)$ and $u(\xi,\tau)$.

4. Deviation of the nonlinear solution from the linear at the very start of the wave process and in the near-front region. On the basis of the assumptions adopted in Sect. 1, the function $\Psi(\tau)$ admits of the representation

$$\Psi(\tau)=\tau\Psi^{\cdot}(0)+{}^1/_2\tau^2\Psi^{\cdot\cdot}(0)+{}^1/_6\tau^3\Psi^{\cdot\cdot\cdot}(0)+\ldots \tag{4.1}$$

as $\tau\to 0$, here $\Psi^{\cdot}(0)$, $\Psi^{\cdot\cdot}(0)$, ... are finite numbers. Let us again consider the case of constant coefficients of representations (1.3) – (1.5). Using (4.1) with small τ, we compute the derivatives and the integrals of $\Psi(\tau-\xi)$ occurring in the formulas in Sect. 3 and construct the jth asymptotic approximation in the form

$$u_j(\xi,\tau)=-\{{}^1/_2(\tau-\xi)^2\varepsilon\Psi^{\cdot}(0)\,[1+\vartheta_{j1}(\xi,\tau)]+$$
$$^1/_6(\tau-\xi)^3\,\varepsilon\Psi^{\cdot\cdot}(0)\,[1+\vartheta_{j2}(\xi,\tau)]+\ldots\}H(\tau-\xi) \tag{4.2}$$

In the case of the zeroth approximation $j=0$

$$\vartheta_{0k}=0\qquad (k=1,2,3,\ldots) \tag{4.3}$$

and representation (4.2) turns into the expansion of linear solution (2.4), while to within the succeeding asymptotic approximations $j=1,2,3,\ldots$

$$\vartheta_{jk}(\xi,\tau)=\sum_{i=1}^{j}\varepsilon^i\sum_{l=0}^{i}(\tau-\xi)^{i-l}\xi^l M_{kil}(\tau-\xi)\quad (k=1,2,3,\ldots) \tag{4.4}$$

Here $M_{kil}(\tau-\xi)$ are polynomials of $(\tau-\xi)^{k-1}$, $(\tau-\xi)^k$, $(\tau-\xi)^{k+1}$, ..., which as $(\tau-\xi)\to 0$ tend to the finite limits

$$\lim_{(\tau-\xi)\to 0} M_{kil}(\tau-\xi)=N_{kil},\qquad N_{kil}=\begin{cases}\text{const} & \text{for}\quad k=1\\ 0 & \text{for}\quad k\geqslant 2\end{cases} \tag{4.5}$$

The numerical values of the coefficients of polynomials $M_{kil}(\tau-\xi)$, including also the values of N_{1il}, are determined by the values of the coefficients of representations (1.3) – (1.5) and of the quantities

$$\frac{\partial^n}{\partial\tau^n}\Psi(\tau)\big|_{\tau=0},\qquad n=k,\ n=k+1,\ n=k+2,\ldots$$

Note that

$$N_{110} = -{}^{1}/_{12}\ (1 + T)\ (A_1 - A_2)\Psi^{\cdot}\ (0),\ \ N_{111} = -{}^{1}/_{2}\ (A_1 - A_2)\Psi^{\cdot}\ (0) \tag{4.6}$$

As $\tau \to 0$, i.e. at the very commencement of the wave process the quantities $\vartheta_{jk}\ (\xi,\ \tau)$ tend to zero in the perturbed domain $0 \leqslant \xi \leqslant \tau$ and for any j and k. Consequently, in the case of the problems under consideration, for sufficiently small values of time τ the solution of the quasi-linear wave equation (1.1) is arbitrarily close to the solution of the linear wave equation (2.1).

Let us now consider the deviation of the solution of Eq. (1.1) from the solution of Eq. (2.1) in the near-front region where the difference $\tau - \xi$ is a small quantity. On the basis of (4.2) – (4.5), as $(\tau - \xi) \to 0$ we have

$$u_j\ (\xi,\ \tau) \sim -{}^{1}/_{2}\ (\tau - \xi)^2 \varepsilon \Psi^{\cdot}\ (0)[1 + \vartheta_j\ (\xi) + O\ (\tau - \xi)]\ H\ (\tau - \xi) \tag{4.7}$$

Here

$$\vartheta_0\,(\xi) = 0 \ \text{ and } \ \vartheta_j = \sum_{i=1}^{j} \varepsilon^i \xi^i N_{1ii} \quad \text{for} \ \ j = 1, 2, 3, \ldots \tag{4.8}$$

In the near-front region ξ increases with a growth of τ. Consequently, for sufficiently large τ the quantities $\vartheta_j(\xi)$ can acquire arbitrarily large values. Hence, for sufficiently large values of time the error in the zeroth (linear) approximation can become arbitrarily large in the near-front region.

On the basis of (3.7) and (4.6) we have

$$N_{111} = {}^{1}/_{2}\,(a_2 - a_1 - b_2 + b_1)\Psi^{\cdot}(0)$$

Let us assume that N_{111} does not equal zero and that $|\ N_{1ii}\ | \leqslant (|\ N_{111}\ |)^i$. Then for sufficiently small τ for which the condition

$${}^{1}/_{2}\,|\,(a_2 - a_1 - b_2 + b_1)\ \varepsilon\xi\ \Psi^{\cdot}\ (0)\ | < 1$$

is fulfilled in the perturbed domain, the zeroth (linear) approximation has an asymptotic error of order

$$\vartheta_1 \sim {}^{1}/_{2}\ \varepsilon\xi\ (a_2 - a_1 - b_2 + b_1)\ \Psi^{\cdot}\ (0) \tag{4.9}$$

in the near-front region. From the coefficients of representations (1.3) – (1.5) only a_1, b_1, a_2 and b_2 occur in estimate (4.9). Consequently, in the case being considered the solution of Eq. (1.1) with coefficients (1.2) – (1.5) can be approximated in the near-front region using the solution of the equation

$$u^{\cdot\cdot}\ (\xi,\ \tau)\ [1 + a_2 u^{\cdot} + a_1 u']\ - u''\ (\xi,\ \tau)\ [1 + b_2 u^{\cdot} + b_1 u'] = 0.$$

5. Deviation of the nonlinear solution from the linear one under one-dimensional transient wave processes of deformation of an elastic half-space. Let us consider the application of the method set forth above to the case of transient wave processes of deformation of an elastic half-space, which in a Cartesian system of Lagrange coordinates depend on the one coordinate X and on time t. Let W be the deformation energy density, referred to a unit volume in the undeformed state, ρ_0 the density in the undeformed state, $U\ (X,\ t)$ the displacement, λ and μ Lame constants, h a constant with the dimension of length. We adopt the notation

$$c = [(\lambda + 2\mu)\ /\ \rho_0]^{1/2}$$

and we introduce dimensionless quantities by the formulas

$$\xi = Xh^{-1}, \quad \tau = cth^{-1}, \quad u = Uh^{-1} \tag{5.1}$$

Then geometrically and physically the nonlinear one-dimensional transient wave process of deformation of an elastic half-space can be described by the equation [6 – 9]

$$u^{\cdot\cdot}(\xi, \tau) - u''(\xi, \tau)\, q(u') = 0 \tag{5.2}$$

where

$$q(u') = \frac{\partial^2 W(e)}{\partial (u')^2}(\lambda + 2\mu)^{-1}, \qquad e = u' + \frac{1}{2}(u')^2 \tag{5.3}$$

Usually the function $W(e)$ is constructed in the form of an expansion [6 – 9] where $q(u')$ has the structure

$$q(u') = 1 + k_1 u'(\xi, \tau) + k_2 [u'(\xi, \tau)]^2 + \ldots \tag{5.4}$$

We remark that the value of coefficient k_1 is determined by the constants of the five-constant theory of elasticity, while even for the computation of k_2 we need more exact information on the physical properties of the meterial.

Equation (5.2) with coefficients (5.4) is a special case of Eq. (1.1) with coefficients (1.2) – (1.5). Using in this special case the formulas for the second asymptotic approximation, constructed in Sect. 3, we have

$$u_2(\xi, \tau) = \Big\{ -\varepsilon \int_0^{\tau-\xi} \Psi'(z)\, dz - \varepsilon^2 \Big[\frac{1}{8}(1 + T) k_1 \int_0^{\tau-\xi} \Psi'^2(z)\, dz + \frac{1}{4} k_1 \xi \Psi'^2(\tau - \xi)\Big] + \varepsilon^3 \Big[\frac{1}{12}(1 + T)\Big(\frac{1}{8}(5 - 3T) k_1^2 - k_2\Big) \int_0^{\tau-\xi} \Psi'^3(z)\, dz + \frac{1}{6}\Big(\frac{1}{8}(5 - 3T) k_1^2 - k_2\Big) \xi \Psi'^3(\tau - \xi) - \frac{1}{24} k_1^2 \xi^2 \frac{\partial}{\partial \tau} \Psi'^3(\tau - \xi)\Big] + \varepsilon^4(0)\Big\} H(\tau - \xi)$$

$$u_2'(\xi, \tau) = \Big\{\varepsilon \Psi'(\tau - \xi) + \varepsilon^2 \Big[-\frac{1}{8}(1 - T) k_1 \Psi'^2(\tau - \xi) + \frac{1}{4} k_1 \xi \frac{\partial}{\partial \tau} \Psi'^2(\tau - \xi)\Big] + \varepsilon^3 \Big[\frac{1}{12}(1 - T)\Big(\frac{1}{8}(5 - 3T) k_1^2 - k_2\Big) \Psi'^3(\tau - \xi) + \Big(\frac{1}{16}(T - 3) k_1^2 + \frac{1}{6} k_2\Big) \xi \frac{\partial}{\partial \tau} \Psi'^3(\tau - \xi) + \frac{1}{24} k_1^2 \xi^2 \frac{\partial^2}{\partial \tau^2} \Psi'^3(\tau - \xi)\Big] + \varepsilon^4(0)\Big\} H(\tau - \xi)$$

$$u_2^{\cdot}(\xi, \tau) = \Big\{-\varepsilon \Psi'(\tau - \xi) - \varepsilon^2 \Big[\frac{1}{8}(1 + T) k_1 \Psi'^2(\tau - \xi) + \frac{1}{4} k_1 \xi \frac{\partial}{\partial \tau} \Psi'^2(\tau - \xi)\Big] + \varepsilon^2 \Big[\frac{1}{12}(1 + T)\Big(\frac{1}{8}(5 - 3T) k_1^2 - k_2\Big) \Psi'^3(\tau - \xi) + \frac{1}{6}\Big(\frac{1}{8}(5 - 3T) k_1^2 - k_2\Big) \xi \frac{\partial}{\partial \tau} \Psi'^3(\tau - \xi) - \frac{1}{24} k_1^2 \xi^2 \frac{\partial^2}{\partial \tau^2} \Psi'^3(\tau - \xi)\Big] + \varepsilon^4(0)\Big\} H(\tau - \xi)$$

$$u_2''(\xi,\tau)=\Big\{-\varepsilon\Psi^{\cdot}(\tau-\xi)+\varepsilon^2\Big[\frac{1}{8}(3-T)k_1\frac{\partial}{\partial\tau}\Psi^2(\tau-\xi)-\frac{1}{4}k_1\xi\frac{\partial^2}{\partial\tau^2}\Psi^2(\tau-\xi)-\varepsilon^2\Big[\frac{1}{12}(3-T)\Big(\frac{1}{8}(7-3T)k_1^2-k_2\Big)+\frac{1}{48}(1+T)k_1^2\Big]\frac{\partial}{\partial\tau}\Psi^3(\tau-\xi)+\varepsilon^2\xi\Big[\frac{1}{6}\Big(\frac{1}{8}(7-3T)k_1^2-k_2\Big)+\frac{1}{8}k_1^2\Big]\frac{\partial^2}{\partial\tau^2}\Psi^3(\tau-\xi)-\varepsilon^3\xi^2\Big[\frac{1}{24}k_1^2\frac{\partial^3}{\partial\tau^3}\Psi^3(\tau-\xi)\Big]+\varepsilon^4(0)\Big\}H(\tau-\xi)$$

$$u_2^{\cdot\cdot}(\xi,\tau)=\Big\{-\varepsilon\Psi^{\cdot}(\tau-\xi)-\varepsilon^3\Big[\frac{1}{8}(1+T)k_1\frac{\partial}{\partial\tau}\Psi^2(\tau-\xi)+\frac{1}{4}\xi k_1\frac{\partial^2}{\partial\tau^2}\Psi^2(\tau-\xi)\Big]+\varepsilon^3\Big[\frac{1}{12}(1+T)\Big(\frac{1}{8}(5-3T)k_1^2-k_2\Big)\frac{\partial}{\partial\tau}\Psi^3(\tau-\xi)\Big]+\varepsilon^3\xi\Big[\frac{1}{6}\Big(\frac{1}{8}(5-3T)k_1^2-k_2\Big)\frac{\partial^2}{\partial\tau^2}\Psi^3(\tau-\xi)\Big]-\varepsilon^3\xi^2\Big[\frac{1}{24}k_1^2\frac{\partial^3}{\partial\tau^3}\Psi^3(\tau-\xi)\Big]+\varepsilon^4(0)\Big\}H(\tau-\xi) \qquad (5.5)$$

Here, as before, $T=1$ for Problem A and $T=-1$ for Problem B. Within the braces in formulas (5.5) the terms with factor ε correspond to the zeroth approximation (the linear solution), while the terms with factors ε^2 and ε^3 are, respectively, the corrections found as a result of computing the first and second asymptotic approximations. Here the terms with factor ε^2 depend on the material's physical constants expressed in k_1, while terms with factor ε^3 are expressed in k_1 and k_2. To compute k_1 we need the constants of the five-constant theory of elasticity, while to compute k_2 we need the constants of a more exact model of the material.

If the conditions

$$\varepsilon|k_1|\ll 1, \qquad |k_2|\lesssim k_1^2 \qquad (5.6)$$

are fulfilled, then formulas (5.5) for computing the derivatives can be simplified, with an asymptotic error of order εk_1, to the following formulas:

$$u_2'(\xi,\tau)=-u_2^{\cdot}(\xi,\tau)=\Big\{\varepsilon\Psi(\tau-\xi)+\frac{1}{4}k_1\varepsilon^2\xi\frac{\partial}{\partial\tau}\Psi^2(\tau-\xi)+\frac{1}{24}k_1^2\varepsilon^3\xi^2\frac{\partial^2}{\partial\tau^2}\Psi^3(\tau-\xi)+\varepsilon^4(0)\Big\}H(\tau-\xi)$$

$$u_2''(\xi,\tau)=u_2^{\cdot\cdot}(\xi,\tau)=\Big\{-\varepsilon\Psi^{\cdot}(\tau-\xi)-\frac{1}{4}k_1\xi\frac{\partial^2}{\partial\tau^2}\Psi^2(\tau-\xi)-\frac{1}{24}k_1^2\varepsilon^3\xi^2\frac{\partial^3}{\partial\tau^3}\Psi^3(\tau-\xi)+\varepsilon^4(0)\Big\}H(\tau-\xi) \qquad (5.7)$$

The solutions of Problems A and B are identical to within formulas (5.7).

6. Example. Suppose that on the surface $X=0$ of an elastic half-space there is applied either the force

$$\partial U(X,t)/\partial t=-\varepsilon c\sin\Omega t\,H(t) \qquad \text{for } X=0 \qquad (6.1)$$

or the force

$$\partial U(X,t)/\partial X=\varepsilon\sin\Omega t\,H(t) \qquad \text{for } X=0 \qquad (6.2)$$

If we introduce the dimensionless quantities (5.1), then in case (6.1) we have Problem B, while in case (6.2) we have Problem A relative to Eq. (5.2). Here

$$\Psi_\cdot(\tau) = \sin(\tau\, h\Omega c^{-1})$$

Let us assume that the specified values of the coefficients k_1, k_2 and ε are such that conditions (5.6) are fulfilled, and let us use the simplified formulas (5.7) of second approximation which with an error of order εk_1 are the same for Problems A and B. We then have

$$\frac{\partial U_2(X,t)}{\partial X} = -\frac{1}{c}\frac{\partial U_2(X,t)}{\partial t} = \Big\{\sin[\Omega(t - Xc^{-1})] + \frac{1}{4}k_1\varepsilon c^{-1}X\frac{\partial}{\partial t}\sin^2[\Omega(t - Xc^{-1})] + \frac{1}{24}k_1^2\varepsilon^2c^{-2}X^2\frac{\partial^2}{\partial t^2}\sin^3[\Omega(t - Xc^{-1})]\Big\}\varepsilon H(t - Xc^{-1}) \tag{6.3}$$

Formula (6.3) can be easily led to the form

$$\frac{\partial U_2(X,t)}{\partial X} = -\frac{1}{c}\frac{\partial U_2(X,t)}{\partial t} = \Big\{\Big(1 - \frac{1}{32}k_1^2\varepsilon^2X^2\Omega^2c^{-2}\Big)\sin[\Omega(t - Xc^{-1})] + \frac{1}{4}k_1\varepsilon X\Omega c^{-1}\sin[2\Omega(t - Xc^{-1})] + \frac{3}{32}k_1^2\varepsilon^2X^2\Omega^2c^{-2}\sin[3\Omega(t - Xc^{-1})]\Big\}\varepsilon H(t - Xc^{-1}) \tag{6.4}$$

To within a linear (zeroth) approximation we have the solution

$$\frac{\partial U_0(X,t)}{\partial X} = -\frac{1}{c}\frac{\partial U_0(X,t)}{\partial t} = \sin[\Omega(t - Xc^{-1})]\,\varepsilon H(t - Xc^{-1}) \tag{6.5}$$

From a comparison of (6.4) and (6.5) if follows that the nonlinear effects increase with the growth of $k_1\varepsilon X\Omega c^{-1}$ and in the case being considered, occur in two forms: a) in the form of a variation in the amplitude of the linear solution, b) in the form of the appearance of higher frequency components of the wave process, which were absent in the linear solution. For sufficiently small t the nonlinear solutions differ arbitrarily little from the linear one, but as time t increases the perturbed region $0 \leqslant X \leqslant tc$ grows and the nonlinear effects become apparent more strongly in that part of the perturbed region where X acquires comparatively large values. For the specified values of k_1, ε, Ω and c the approximation method used is suitable in that part of the perturbed region where $k_1\varepsilon X\Omega c^{-1}$ is less than or of the order of unity.

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CONTINUAL MECHANOCHEMICAL MODEL OF MUSCULAR TISSUE

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The behavior of active muscular tissue is described with the help of a closed system of equations of motion of a two-phase, multicomponent, anisotropic continuous medium, with the mechanochemical processes occurring within it taken into account. The fundamental hypotheses are based on the information of general character concerning the structure and performance of the muscular tissue. It is assumed that the phase in which the mechanochemical reactions take place is viscoelastic, while the other phase is assumed elastic. The medium is assumed to have single velocity, although a passage of components between the phases is allowed. The laws of conservation are given and the rheological equations are written in accordance with the general principles of the mechanics of continuous medium and thermodynamics of irreversible processes [1 – 4]. It is shown that the model constructed describes, e. g., such characteristic properties of the muscle tissue as the existence of stresses in the absence of strains, zero-load deformations, and dissipation of energy in the state of mechanical equilibrium.

The activity of the muscular tissue is governed by chemical processes taking place in the tissue, within the specific ordered structures called myofibrillae and, in the final count, by the mechanochemical reactions which affect the form or the relative distribution of the protein molecules [5 – 8]. Outside the myofibrillae we have various auxilliary systems, the connecting tissue and other structures, including capillary blood vessels which serve as the source of initial chemical compounds. The onset of active muscular contraction is connected with the arrival of specific reagents at the myofibrillae.

The study of various physiological phenomena (such as the working of the